МАТЕМАТИКА

УДК 517.925

О ПЕРИОДИЧЕСКИХ РЕШЕНИЯХ РАЦИОНАЛЬНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

Чжиньсинь Чжоу

Янчжоуский университет, Янчжоу, Китай

ON THE PERIODIC SOLUTIONS OF THE RATIONAL DIFFERENTIAL EQUATIONS

Zhengxin Zhou

Yangzhou University, Yangzhou, China

В работе используется метод Мироненко для изучения периодических рациональных дифференциальных уравнений. Полученные результаты применяются для получения достаточных условий центра особых точек полиномиальных дифференциальных систем.

***Ключевые слова**: отражающая функция; условия центра; периодические решения.*

In this paper Mironenko method to study the periodic solutions of the rational differential equations is used. The obtained results to derive the sufficient conditions for a critical point of some polynomial differential systems to be a center are applied.

***Keywords**: reflecting function; center conditions; periodic solution.*

AMS subject classifications: 34A12.

Introduction

This paper deals with the qualitative behavior of solutions of rational differential equation

$$\frac{dr}{d\theta}=\frac{a_0+a_1r+a_2r^2+a_3r^3+a_4r^4}{b_0+b_1r+r^2}:=X(\theta,r) \quad (0.1)$$

which has a continuous differentiable right-hand side and a unique solution for the initial value problem,

$$a_i:=a_i(\theta),\ i=0,1,2,3,4;\ b_j:=b_j(\theta),\ j=0,1.$$

The main reason of great interest in these equations is that they are closely related to planar vector fields. It is known [1]–[3], for polynomial differential system

$$\begin{cases}\dfrac{dx}{dt}=\sum\limits_{i+j=1}^{n+1}p_{ij}x^iy^j,\\[2mm] \dfrac{dy}{dt}=\sum\limits_{i+j=1}^{n+1}q_{ij}x^iy^j,\end{cases} \quad (0.2)$$

where p_{ij}, q_{ij} are real constants. There has been a longstanding problem, called the Poincaré center-focus problem, for the system (0.2) one can find explicit conditions of p_{ij}, q_{ij} under which (0.2) has a center at the origin (0, 0), i. e., all the orbits nearby are closed. This problem is equivalent to an analogue for a corresponding periodic equation

$$\frac{dr}{d\theta}=\frac{\sum\limits_{i=0}^{n}a_i(\theta)r^{i+1}}{\sum\limits_{i=0}^{n}b_i(\theta)r^i}. \quad (0.3)$$

To see this let us note that the phase curves of (0.2) near the origin (0,0) in polar coordinates $x=r\cos\theta,\ y=r\sin\theta$ are determined by (0.3), where $a_i(\theta), b_i(\theta), i=0,1,2,\dots,n$ are polynomials in $\cos\theta,\ \sin\theta.$

The limit cycles of (0.2) correspond to 2π-periodic solutions of (0.3). The planar vector field (0.2) has a center at (0,0) if and only if the equation (0.3) has a center at $r=0$, i. e., all the solutions nearby are 2π-periodic [1]–[3].

The method of Lyapunov is often used to study the center-focus problem, but for high-order systems it is very difficult to give the center conditions. In this paper the method of reflecting function to study the behavior of solutions of (0.1) with the sufficient conditions for $r = 0$ to be a center is applied.

Now the concept of the reflecting function, which will be used throughout the rest of this article is introduced.

1 The main facts from the theory of reflecting function

Consider differential system

$$\frac{dx}{dt}=X(t,x),\ t\in R,\ x\in R^n \quad (1.1)$$

which has a continuous differentiable right-hand side and general solution $\varphi(t;t_0,x_0)$.

For each such system, the reflecting function is defined (see [4]) as $F(t,x):=\varphi(-t;t,x)$. Therefore, for any solution $x(t)$ of (1.1)

$$F(t,x(t))=x(-t),\ F(0,x)=0.$$

If the system (1.1) is 2ω-periodic with respect to t, and $F(t,x)$ is its reflecting function, then

$$T(x) := F(-\omega, x) = \varphi(\omega; -\omega, x)$$

is the Poincaré mapping of (1.1) over the period $[-\omega,\omega]$. Thus, the solution $x=\varphi(t;-\omega,x_0)$ of (1.1) defined on $[-\omega,\omega]$ is 2ω-periodic if and only if x_0 is a fixed point of $T(x)$. The stability of this periodic solution is equivalent to the stability of the fixed point x_0.

A differentiable function $F(t,x)$ is a reflecting function of the system (1.1) if and only if it is a solution of the Cauchy problem

$$F_t + F_x X(t,x) + X(-t,F) = 0, \quad F(0,x) = x.$$

There are many papers which are also devoted to the investigations of qualitative behavior of solutions of differential systems with the help of reflecting functions [4]–[14].

2 Reflecting function and periodic solutions of the equation

Let us consider differential equation (0.1).

***Theorem* 2.1.** *Suppose that functions*

$$\alpha_0 := a_4,$$

$$\alpha_1 := a_3 - a_4 b_1,$$

$$\alpha_2 := a_2 - a_3 b_1 - 2a_4 b_0 + \frac{1}{4} a_4 b_1^2$$

are odd functions and

$$a_1 + a_3 b_0 + \frac{1}{2} a_3 b_1^2 - a_2 b_1 - 2a_4 b_1 b_0 + \frac{1}{2} b_1 b_1{}' - b_0{}' = 0;$$

$$a_0 + \frac{1}{2} a_3 b_1 b_0 + \frac{1}{8} a_3 b_1^2 - a_2 b_1^2 - \frac{1}{2} a_4 b_0 b_1^2 - $$

$$-a_4 b_0^2 - \frac{1}{2} b_1 b_1{}' + \frac{1}{2} b_0 b_1{}' + \frac{1}{8} b_1^2 b_1{}' = 0.$$

Then the reflecting function $F(\theta,r)$ *of equation* (0.1) *satisfies the following relation*

$$F + \frac{\overline{b}_1^2 - 4\overline{b}_0}{4F + 2\overline{b}_1} = r + \frac{b_1^2 - 4b_0}{4r + 2b_1}, \qquad (2.1)$$

where $\overline{b}_0 := b_0(-\theta)$, $\overline{b}_1 := b_1(-\theta)$.

Proof. To prove the present result one only needs to check that the function F implied from (2.1), satisfies

$$F_\theta + F_r X(\theta,r) + X(-\theta,F) = 0, \quad F(0,r) = 0. \qquad (2.2)$$

Let

$$\Phi := F + \frac{\overline{b}_1^2 - 4\overline{b}_0}{4F + 2\overline{b}_1} - r - \frac{b_1^2 - 4b_0}{4r + 2b_1},$$

then

$$F_\theta = -\frac{\Phi_\theta}{\Phi_F}, \; F_r = -\frac{\Phi_r}{\Phi_F}.$$

Substituting it into (2.2)

$$\frac{(\overline{b}_1^2 - 4\overline{b}_0)'(4F + 2\overline{b}_1) - 2\overline{b}_1{}'(\overline{b}_1^2 - 4\overline{b}_0)}{(4F + 2\overline{b}_1)^2} -$$

$$-\frac{(b_1^2 - 4b_0)'(4r + 2b_1) - 2b_1{}'(b_1^2 - 4b_0)}{(4r + 2b_1)^2} =$$

$$= \left(1 - \frac{4(\overline{b}_1^2 - 4\overline{b}_0)}{(4F + 2\overline{b}_1)^2}\right)\overline{X} - \left(\frac{4(b_1^2 - 4b_0)}{(4r + 2b_1)^2} - 1\right)X,$$

where $\overline{X} := X(-\theta, F)$.

On the other hand, by simple computation

$$\left(\frac{4(b_1^2 - 4b_0)}{(4r + 2b_1)^2} - 1\right)X -$$

$$-\frac{(b_1^2 - 4b_0)'(4r + 2b_1) - 2b_1'(b_1^2 - 4b_0)}{(4r + 2b_1)^2} =$$

$$= -\left(\alpha_0 + \alpha_1\left(r + \frac{b_1^2 - 4b_0}{4r + 2b_1}\right) + \alpha_2\left(r + \frac{b_1^2 - 4b_0}{4r + 2b_1}\right)^2\right),$$

$$\left(1 - \frac{4(\overline{b}_1^2 - 4\overline{b}_0)}{(4F + 2\overline{b}_1)^2}\right)\overline{X} -$$

$$-\frac{(\overline{b}_1^2 - 4\overline{b}_0)'(4F + 2\overline{b}_1) - 2\overline{b}_1'(\overline{b}_1^2 - 4\overline{b}_0)}{(4F + 2\overline{b}_1)^2} =$$

$$= \overline{\alpha}_0 + \overline{\alpha}_1\left(F + \frac{\overline{b}_1^2 - 4\overline{b}_0}{4F + 2\overline{b}_1}\right) + \overline{\alpha}_2\left(F + \frac{\overline{b}_1^2 - 4\overline{b}_0}{4F + 2\overline{b}_1}\right)^2.$$

Thus the identity (2.2) becomes

$$\alpha_0 + \alpha_1\left(r + \frac{b_1^2 - 4b_0}{4r + 2b_1}\right) + \alpha_2\left(r + \frac{b_1^2 - 4b_0}{4r + 2b_1}\right)^2 +$$

$$+\overline{\alpha}_0 + \overline{\alpha}_1\left(F + \frac{\overline{b}_1^2 - 4\overline{b}_0}{4F + 2\overline{b}_1}\right) + \overline{\alpha}_2\left(F + \frac{\overline{b}_1^2 - 4\overline{b}_0}{4F + 2\overline{b}_1}\right)^2 = 0.$$

As $\Phi = 0$, thus the identify (2.2) is correct, i. e., the function $F(\theta,r)$ implied from (2.1) is a reflecting function of (0.1). The proof is finished.

***Corollary* 2.1**. *Suppose that all the conditions of Theorem* 2.1 *are satisfied, and* $a_i(\theta), b_j(\theta)$ $i = 0,1,2...,4,\ j = 0,1$ *are* 2π*-periodic functions, then all the solutions of* (0.1) *defined on* $[-\pi,\pi]$ *are* 2π*-periodic.*

Proof. From the above assumptions it follows that the reflecting function $F(\theta,r)$ of (0.1) satisfies the identity (2.1) and is 2π-periodic, then by [4], all the solutions of (0.1) defined on $[-\pi,\pi]$ are 2π-periodic.

Now let us rewrite (0.1) in the following form

$$\frac{dr}{d\theta} = c_0 + c_1 r + a_4 r^2 + \frac{d_0 + d_1 r}{b_0 + b_1 r + r^2} := Y(\theta, r), \quad (2.3)$$

where

$$c_1 := a_3 - a_4 b_1,$$

$$c_0 := a_2 - a_4 b_0 - b_1 c_1,$$

$$d_1 := a_1 - b_0 c_1 - b_1 c_0,$$

$$d_0 := a_0 - b_0 c_0.$$

***Theorem* 2.2** *Suppose that*

$$\lim_{t\to 0}\frac{a_4}{\overline{a}_4} = -1; \quad a_4{}^2 d_1 + \overline{a}_4{}^2 \overline{d}_1 = 0;$$

$$\overline{d}_0 + d_0 f_1^3 - \overline{b}_1\overline{d}_1 + b_1\overline{d}_1 f_1 - \overline{d}_1 f_0 = 0;$$

$$d_0\overline{b}_1 f_1^2 + \overline{d}_0 b_1 + b_1\overline{d}_1 f_0 + d_1\overline{b}_1 f_0 f_1 +$$
$$+(b_0\overline{d}_1 + \overline{b}_0 d_1) f_1 + 2d_0 f_0 f_1^2 + d_1 f_0{}^2 f_1 = 0;$$

$$d_0\overline{b}_0 f_1 + b_0\overline{d}_0 + d_0\overline{b}_1 f_0 f_1 + d_0 f_0{}^2 f_1 + b_0\overline{d}_1 f_0 = 0;$$

$$f_0{}' + \overline{c}_0 + \overline{c}_1 f_0 + \overline{a}_4 f_0{}^2 + c_0 f_1 = 0,$$

where

$$f_1 = -\frac{a_4}{\overline{a}_4},\; f_0 = \frac{f_1{}' + (c_1 + \overline{c}_1) f_1}{2a_4}.$$

Then $F(\theta, r) = f_0 + f_1 r$ *is the reflecting function of equation* (2.3).

In addition, if $a_i(\theta), b_j(\theta)$ $i = 0,1,2...,4$, $j = 0,1$ are 2π-periodic functions, then all the solutions of (2.3) defined on $[-\pi,\pi]$ are 2π-periodic.

Proof. Using the above conditions, it is not difficult to check that $F(\theta, r) = f_0 + f_1 r$ is the solution of the Cauchy problem

$$F_\theta + F_r\,\Upsilon(\theta, r) + \Upsilon(-\theta, F) = 0,$$
$$F(0, r) = 0.$$

Therefore, $F(\theta, r) = f_0 + f_1 r$ is the reflecting function of (2.3). On the other hand, as $a_4(\theta), c_1(\theta)$ are 2π-periodic, then $F(\theta, r) = f_0 + f_1 r$ is 2π-periodic, so all the solutions of (2.3) defined on $[-\pi,\pi]$ are 2π-periodic.

Now consider the equation

$$\frac{dr}{d\theta} = \frac{a_2 r^2 + a_3 r^3 + a_4 r^4}{1 + b_1 r + b_2 r^2} := Z(\theta, r) \qquad (2.4)$$

which has a continuous differentiable right-hand side and posesses a unique solution for the initial value problem,

$$a_i := a_i(\theta), b_j := b_j(\theta)\ (i = 2,3,4, j = 1,2).$$

Denoting:

$$\delta_1 := a_3 - a_2 b_1,\; \delta_2 := a_4 - a_2 b_2,$$
$$\alpha := \int_0^\theta (a_2(s) + a_2(-s))ds.$$

***Theorem* 2.3.** *Suppose that* δ_1 *is an odd function and*

$$\alpha\delta_1 + \delta_2 + \overline{\delta}_2 + b_1\overline{\delta}_1 + \overline{b}_1\delta_1 = 0;$$
$$\alpha(\delta_2 - \overline{\delta}_2) + \overline{b}_1\delta_2 + b_1\overline{\delta}_2 + \overline{b}_2\delta_1 + b_2\overline{\delta}_1 = 0;$$
$$\alpha^2(\delta_2 + \overline{\delta}_2) + \alpha(\overline{b}_1\delta_2 \text{-} b_1\overline{\delta}_2 + b_2\overline{\delta}_1 - \overline{b}_2\delta_1) +$$
$$+2\overline{b}_2\delta_2 + 2b_2\overline{\delta}_2 = 0.$$

Then $F = \dfrac{r}{1+\alpha r}$ *is the reflecting function of* (2.4). *Besides, if* $a_i(\theta), b_j(\theta)$ $i = 0,1,2...,4$, $j = 0,1$ *are* 2π*-periodic functions, then all the solutions of* (2.4) *defined on* $[-\pi,\pi]$ *are* 2π*-periodic.*

Proof. Using the above assumptions, it is not difficult to check that $F = \dfrac{r}{1+\alpha r}$ is the solution of the Cauchy problem

$$F_\theta + F_r\,Z(\theta, r) + Z(-\theta, F) = 0,$$
$$F(0, r) = 0.$$

Thus, $F = \dfrac{r}{1+\alpha r}$ is the reflecting function of (2.4) and 2π-periodic, therefore, the present result is true.

3 Example

Consider polynomial differential system:

$$\begin{cases} x' = -y + P_2 + P_3 + \\ \quad + x(p_{40}x^3 + p_{31}x^2 y + p_{22}xy^2 + p_{03}y^3), \\ y' = x + Q_2 + Q_3 + \\ \quad + y(p_{40}x^3 + p_{31}x^2 y + p_{22}xy^2 + p_{03}y^3), \end{cases} \qquad (3.1)$$

where

$$P_k = \sum_{i+j=k} p_{ij}x^i y^j,$$
$$Q_k = \sum_{i+j=k} q_{ij}x^i y^j,$$
$$p_{ij}, q_{ij}\ (k = 2,3)$$

are constants.

Taking $x = r\cos\theta$, $y = r\sin\theta$, the system (3.1) becomes to (2.4) with

$$a_2 = p_{20}C^3 + (p_{11} + q_{20})C^2 S + (p_{02} + q_{11})CS^2 + q_{02}S^3;$$
$$a_3 = p_{30}C^4 + (p_{21} + q_{30})C^3 S +$$
$$+(p_{12} + q_{21})C^2 S^2 + (p_{03} + q_{12})CS^3 + q_{03}S^4;$$
$$a_4 = p_{40}C^3 + \mathrm{p}_{31}C^2 S + p_{13}S^3 + p_{22}CS^2;$$
$$b_1 = q_{20}C^3 + (q_{11} - p_{20})C^2 S + (q_{12} - p_{11})CS^2 - p_{02}S^3;$$
$$b_2 = q_{30}C^4 + (q_{21} - p_{30})C^3 S +$$
$$+(q_{12} - p_{21})C^2 S^2 + (q_{03} - p_{12})CS^3 - p_{03}S^4,$$

where $C := \cos\theta$, $S := \sin\theta$.

If all the conditions of Theorem 2.3 are satisfied, then the origin point (0,0) of system (3.1) is the center.

Conclusion

The paper shows that, applying Mironenko method for reflecting function, one can easily get the center conditions for the above system. While using Lyapunov method, obviously, it is difficult to determine when the origin (0,0) is a center. Therefore, sometimes it is better to solve the center-focus problem using the method of reflecting function than the method of Lyapunov.

REFERENCES

1. *Alwash*, *M.A.M.* Non-autonomous equations related to polynomial two-dimensional systems / M.A.M. Alwash, N.G. Lloyd // Proc. Roy. Soc. Edinburgh Sect. – 1987. – Vol. A 105. – P. 129–152.

2. *Yang*, *Lijun.* Some new results on Abel equations / Yang Lijun, Tang Yuan // J. Math. Anal. Appl. – 2001. – Vol. 261. – P. 100–112.

3. *Sadowski*, *A.P.* Polynomial ideals and manifold / A.P. Sadowski // Minsk : University Press, 2008.

4. *Mironenko*, *V.I.* Analysis of reflective function and multivariate differential system / *V.I. Mironenko.* – Gomel : University Press, 2004. – 196 p.

5. *Arnold*, *V.I.* Ordinary differential equation / V.I. Arnold. – Moscow : Science Press, 1971. – P. 198–240.

6. *Mironenko*, *V.I.* The reflecting function of a family of functions / V.I. Mironenko // Differ. Equ. – 2000. – Vol. 36, № 12. – P. 1636–1641.

7. *Alisevich*, *L.A.* On linear system with triangular reflective function / L.A. Alisevich // Differ. Equ. – 1989. – Vol. 25, №3. – P. 1446–1449.

8. *Musafirov*, *E.V.* Differential systems, the mapping over period for which is represented by a product of three exponential matrixes / E.V. Musafirov // J. Math. Anal. Appl. – 2007. – Vol. 329. – P. 647–654.

9. *Mironenko*, *V V.* Time symmetry preserving perturbations of differential systems / V.V. Mironenko // Differ. Equ. – 2004. – Vol. 40, №20. – P. 1395–1403.

10. *Verecovich*, *P.P.* Nonautonomous second order quadric system equivalent to linear system / P.P. Verecovich // Differ. Equ. – 1998. – Vol. 34, №12. – P. 2257–2259.

11. *Maiorovskaya*, *S.V.* Quadratic systems with a linear reflecting function / *S.V. Maiorovskaya* // Differ. Equ. – 2009. – Vol. 45, № 2. – P. 271–273.

12. *Zhengxin*, *Zhou.* On the reflective function of polynomial differential system / Zhou Zhengxin // J. Math. Anal. Appl. – 2003. – Vol. 278, № 1. – P. 18–26.

13. *Zhengxin*, *Zhou.* The structure of reflective function of polynomial differential systems / Zhou Zhengxin // Nonlinear Analysis. – 2009. – Vol. 71. – P. 391–398.

14. *Zhengxin*, *Zhou*. Research on the properties of some planar polynomial differential equations / Zhou Zhengxin // Appl. Math. Comput. – 2012. – Vol. 218. – P. 5671–5681.

Поступила в редакцию 14.11.13.